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Hence triangle $ECC' <$ triangle EBB'. But, $ABCD = AB'ECD + ABB' + EBB'$ and $AB'C'D = AB'ECD + DCC' + ECC'$.

Hence $ABCD$ is greater than $AB'C'D$.

COROLLARY. It follows, that the quadrilateral of three equal sides, and maximum area, is a trapezoid; that the angles including the fourth side are also equal; that the opposite angles are supplementary; and that the trapezoid is inscriptible.

[*To be continued.*]

PROFESSOR SYLVESTER'S RECIPROCANTS.

By F. P. MATZ, M. Sc., Ph. D., New Windsor, Maryland.

To those functions of the successive derivatives of y with respect to x, which preserve their form unaltered, except for dy/dx as a factor, when the independent and dependent variables x and y are interchanged, Professor Sylvester gave the name of *Reciprocants*.

According to the general theory with respect to the inversion of the indepent and dependent variable, we must have the relations:

$$\frac{dy}{dx}=1\Big/\frac{dx}{dy};\ \frac{d^2y}{dx^2}=\frac{d}{dx}\left(1\Big/\frac{dx}{dy}\right)=\frac{d}{dy}\left(1\Big/\frac{dx}{dy}\right)\frac{dy}{dx}=-\frac{d^2x}{dy^2}\Big/\left(\frac{dx}{dy}\right)^3;$$

$$\frac{d^3y}{dx^3}=\frac{d}{dy}\left[-\frac{d^2x}{dy^2}\Big/\left(\frac{dx}{dy}\right)^3\right]\frac{dy}{dx}=-\left[\frac{dx}{dy}\cdot\frac{d^3x}{dy^3}-3\left(\frac{d^2x}{dy^2}\right)^2\right]\Big/\left(\frac{dx}{dy}\right)^5;$$

$$\frac{d^4y}{dx^4}=-\left[\left(\frac{dx}{dy}\right)^2\left(\frac{d^4x}{dy^4}\right)-10\frac{dx}{dy}\cdot\frac{d^2x}{dy^2}\cdot\frac{d^3x}{dy^3}+15\left(\frac{d^2x}{dy^2}\right)^3\right]\Big/\left(\frac{dx}{dy}\right)^7;\text{ etc.}$$

After these relations are substituted for the various differential coefficients of y with respect to x, in any function of these differential coefficients or derivatives, we are said to have interchanged the independent and dependent variable.

Assume $dy/dx=T$, $d^2y/dx^2=A\,\underline{|2}$, $d^3y/dx^3=B\,\underline{|3}$, $d^4y/dx^4=C\,\underline{|4}$, etc.; then, after eliminating the constants in the general equation of the straight line, by the method of differentiation, we obtain $d^2y/dx^2=A,=d^2x/dy^2=0$ ....(1).

The left-hand member of (1) is Professor Sylvester's first *pure* reciprocant, since it does not involve dy/dx; and this reciprocant is briefly and typically expressed by A. The third member of (1) represents the reciprocant when the independent and dependent variables x and y are interchanged.

The equation of the parobola $(\alpha x+\beta y)^2+2gx+2fy+c=0$, in which $\alpha^2=a$ and $\beta^2=b$, gives the second pure reciprocant:

$$3\frac{d^2y}{dx^2}\cdot\frac{d^4y}{dx^4}-5\left(\frac{d^3y}{dx^3}\right)^2=4AC-5B^2,=3\frac{d^2x}{dy^2}\cdot\frac{d^4x}{dy^4}-5\left(\frac{d^3x}{dy^3}\right)^2=0\ldots.(2).$$

The general equation of a conic, in Cartesian co-ordinates, leads to the pure reciprocant:

$$9\left(\frac{d^2y}{dx^2}\right)^2\frac{d^5y}{dx^5}-45\frac{d^2y}{dx^2}\cdot\frac{d^3y}{dx^3}\cdot\frac{d^4y}{dx^4}+40\left(\frac{d^3y}{dx_2}\right)^3=A^2D-3ABC+2B^3,$$

$$=9\left(\frac{d^2x}{dy^2}\right)^2\frac{d^5x}{dy^5}-45\frac{d^2x}{dy^2}\cdot\frac{d^3x}{dy^2}\cdot\frac{d^4x}{dy^4}+40\left(\frac{d^3x}{dy^4}\right)=0\ldots.(3),$$ which is appropriately denominated the *Mongian Reciprocant.*

Assume $xy=c$; then $xdy/dx+y=0$, and

$$x=-2\frac{dy}{dx}\bigg/\frac{d^2y}{dx^2}.\quad\therefore\ 2\frac{dy}{dx}\cdot\frac{d^3y}{dx^3}-3\left(\frac{d^2y}{dx^2}\right)^2=BT-A^2,=2\frac{dx}{dy}\cdot\frac{d^3x}{dy^3}$$

$$-3\left(\frac{d^2x}{dy^2}\right)^2=0\ldots.(4),$$ which is probably the simplest type of *mixed* reciprocant. From the equation of the hyperbola, $xy+ax+by+c=0$, we can also deduce the *Schwartzian Reciprocant* represented by (4); and after dividing (4) by $(dy/dx)^2$, we have Professor Cayley's *Schwartzian Derivative,*

$$\frac{d^3y/dx^3}{dy/dx}-\frac{3}{2}\left(\frac{d^2y/dx^2}{dy/dx}\right)^3=0,$$ which is directly deducible from $y=(ax+b)$ $(Ax+B)$, and is of practical use in the solution of differential equations.

From the equation of the circle, $x^2+y^2=r^2$, we have $y(dy/dc)+x=0$;

$$\left(\frac{dy}{dx}\right)^2+y\frac{d^2y}{dx^2}+1=0;\ 3\frac{dy}{dx}\cdot\frac{d^2y}{dx^2}\bigg/\frac{d^3y}{dx^3}+y=0;\ 3\left(\frac{d^2y}{dx^2}\right)^2\frac{d^3y}{dx^3}$$

$$-3\frac{dy}{dx}\cdot\frac{d^2y}{dx^2}\cdot\frac{d^4y}{dx^4}+4\left(\frac{d^3y}{dx^3}\right)^2\frac{dy}{dx}=A^2B-2(AC-B^2)T,$$

$$=3\left(\frac{d^2x}{dy^2}\right)^2\frac{d^3x}{dy^3}-3\frac{dx}{dy}\cdot\frac{d^2x}{dy^2}\cdot\frac{d^4x}{dy^4}+4\left(\frac{d^3x}{dy^3}\right)^2\frac{dx}{dy}=0\ldots.(5).$$

The general equation of the circle, after three differentiations, gives

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]\frac{d^3y}{dx^3}-3\left(\frac{d^2y}{dx^2}\right)^2\frac{dy}{dx}=(1+T^2)B-2A^2T,$$

$$=\left[1+\left(\frac{dx}{dy}\right)^2\right]\frac{d^3x}{dy^3}-3\left(\frac{d^2x}{dy^2}\right)^2\frac{dx}{dy}=0\ldots.(6).$$

The reciprocant represented by (6) may be written

$$1+\left(\frac{dy}{dx}\right)^2-3\left(\frac{d^2y}{dx^2}\right)^2\frac{dy}{dx}\Big/\frac{d^3y}{dx^3}=0\ldots(\alpha).$$

After differentiating (α), etc., we have the reciprocant represented by (5).

From the equation $x^2+xy+y^2=1$ may be deduced by differentiation, etc.,

$$x, =-\left(\frac{2dy+dx}{dy+2dx}\right)y, =-\left(\frac{2(dy/dx)+1}{dy/dx+2}\right)y\ldots(a);$$

$$\left(\frac{dy}{dx}+2\right)^2=-\left(2\frac{dy}{dx}+1\right)\left(\frac{dy}{dx}+2\right)\frac{dy}{dx}-3y\frac{d^2y}{dx^2}\ldots(b);$$

$$3y=-\left[2\left(\frac{dy}{dx}\right)^3+6\left(\frac{dy}{dx}\right)^2+6\left(\frac{dy}{dx}\right)+4\right]\Big/\frac{d^2y}{dx^2}\ldots(c).$$

$$\therefore 6\left(\frac{dy}{dx}\right)^2\left(\frac{d^2y}{dx^2}\right)^2+15\left(\frac{d^2y}{dx^2}\right)^2\frac{dy}{dx}+6\left(\frac{d^2y}{dx^2}\right)^2-2\left(\frac{dy}{dx}\right)^3\frac{d^3y}{dx^3}-6\left(\frac{dy}{dx}\right)^2\frac{d^3y}{dx^3}$$

$$-6\frac{dy}{dx}\cdot\frac{d^3y}{dx^3}-4\frac{d^3y}{dx^3}, =T^3-\left(\frac{2A^2-3B}{B}\right)T^2-\left(\frac{5A^2-3B}{B}\right)T-2\left(\frac{A^2-B}{B}\right), =0$$

....(7), which is a *cubic* reciprocant. From (6) is deduced $T^2-2(A^2/B)T+1=0$, which is a *quadratic* reciprocant. Transforming (5), we have $T=A^2B/2(AC-B^2)$;$=A^2/B$,$=5AB/2C$, = etc., which are *linear* reciprocants. In so far as their number is concerned, the pure reciprocants are like the major planets of the solar system—few; while the mixed reciprocants are like the minor planets of the same system—many.

NOTE.—Since we had the good fortune to be one of Professor Sylvester's students—one of the last and probably the youngest he had—in the Johns Hopkins University, we declare that Dr. Halsted's biographical sketch of that *enthusiastic* mathematical professor and investigator is the acme of appropriateness; and we also declare that all the commendation which that sketch accords to Professor Sylvester, is fully merited by him.

IS THE SUPPLEMENTARY ANGLE FINITE OR NOT?

By Professor JOHN N. LYLE, Ph. D., Westminster College, Fulton, Missouri.

If any individual angle whatever is greater than one right angle and